Ejercicios Teoría Cuántica de Campos. Capítulo 42

Autor del curso: Javier García

Problemas resueltos por: Roger Balsach

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1. Calcular la acción de $S[\Lambda]$ sobre la base de espinores.

Dada la matriz

$$S[\Lambda] = \begin{pmatrix} \cosh\frac{\eta}{2} & 0 \\ 0 & \cosh\frac{\eta}{2} \end{pmatrix} \otimes \begin{pmatrix} e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2} & -e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2} \\ e^{i\frac{\phi}{2}}\sin\frac{\theta}{2} & e^{i\frac{\phi}{2}}\cos\frac{\theta}{2} \end{pmatrix} + \begin{pmatrix} 0 & \sinh\frac{\eta}{2} \\ \sinh\frac{\eta}{2} & 0 \end{pmatrix} \otimes \begin{pmatrix} e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2} & e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2} \\ e^{i\frac{\phi}{2}}\sin\frac{\theta}{2} & -e^{i\frac{\phi}{2}}\cos\frac{\theta}{2} \end{pmatrix}$$

Queremos calcular su acción sobre la base de espinores $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. Esta base se puede expresar como el producto tensorial de la siguiente forma

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

De modo que

$$\begin{aligned} S[\Lambda]\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} \cosh\frac{\eta}{2} & 0 \\ 0 & \cosh\frac{\eta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2} & -e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2} \\ e^{i\frac{\phi}{2}}\sin\frac{\theta}{2} & e^{i\frac{\phi}{2}}\cos\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & \sinh\frac{\eta}{2} \\ \sinh\frac{\eta}{2} & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2} & e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2} \\ e^{i\frac{\phi}{2}}\sin\frac{\theta}{2} & -e^{i\frac{\phi}{2}}\cos\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \cosh\frac{\eta}{2} \\ 0 \end{pmatrix} \otimes \begin{pmatrix} e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2} \\ e^{i\frac{\phi}{2}}\sin\frac{\theta}{2} \end{pmatrix} + \begin{pmatrix} 0 \\ \sinh\frac{\eta}{2} \end{pmatrix} \otimes \begin{pmatrix} e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2} \\ e^{i\frac{\phi}{2}}\sin\frac{\theta}{2} \end{pmatrix} \\ &= \boxed{\begin{pmatrix} \cosh\frac{\eta}{2} \\ \sinh\frac{\eta}{2} \end{pmatrix} \otimes \begin{pmatrix} e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2} \\ e^{i\frac{\phi}{2}}\sin\frac{\theta}{2} \end{pmatrix}} \end{aligned} \tag{1}$$

Para el segundo vector hacemos algo muy similar;

$$\begin{aligned} S[\Lambda]\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} \cosh\frac{\eta}{2} & 0 \\ 0 & \cosh\frac{\eta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2} & -e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2} \\ e^{i\frac{\phi}{2}}\sin\frac{\theta}{2} & e^{i\frac{\phi}{2}}\cos\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & \sinh\frac{\eta}{2} \\ \sinh\frac{\eta}{2} & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2} & e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2} \\ e^{i\frac{\phi}{2}}\sin\frac{\theta}{2} & -e^{i\frac{\phi}{2}}\cos\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \cosh\frac{\eta}{2} \\ 0 \end{pmatrix} \otimes \begin{pmatrix} -e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2} \\ e^{i\frac{\phi}{2}}\cos\frac{\theta}{2} \end{pmatrix} + \begin{pmatrix} 0 \\ \sinh\frac{\eta}{2} \end{pmatrix} \otimes \begin{pmatrix} e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2} \\ -e^{i\frac{\phi}{2}}\cos\frac{\theta}{2} \end{pmatrix} \\ &= \boxed{\begin{pmatrix} \cosh\frac{\eta}{2} \\ -\sinh\frac{\eta}{2} \end{pmatrix} \otimes \begin{pmatrix} -e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2} \\ e^{i\frac{\phi}{2}}\cos\frac{\theta}{2} \end{pmatrix}} \end{aligned} \tag{2}$$

Y lo mismo para los otros dos;

$$
\begin{aligned}
S[\Lambda]\begin{pmatrix}0\\0\\1\\0\end{pmatrix} &= \begin{pmatrix}\cosh\frac{\eta}{2} & 0\\ 0 & \cosh\frac{\eta}{2}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2} & -e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2}\\ e^{i\frac{\phi}{2}}\sin\frac{\theta}{2} & e^{i\frac{\phi}{2}}\cos\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\\
&+\begin{pmatrix}0 & \sinh\frac{\eta}{2}\\ \sinh\frac{\eta}{2} & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2} & e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2}\\ e^{i\frac{\phi}{2}}\sin\frac{\theta}{2} & -e^{i\frac{\phi}{2}}\cos\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\\
&=\begin{pmatrix}0\\ \cosh\frac{\eta}{2}\end{pmatrix}\otimes\begin{pmatrix}e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2}\\ e^{i\frac{\phi}{2}}\sin\frac{\theta}{2}\end{pmatrix}+\begin{pmatrix}\sinh\frac{\eta}{2}\\ 0\end{pmatrix}\otimes\begin{pmatrix}e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2}\\ e^{i\frac{\phi}{2}}\sin\frac{\theta}{2}\end{pmatrix}\\
&=\boxed{\begin{pmatrix}\sinh\frac{\eta}{2}\\ \cosh\frac{\eta}{2}\end{pmatrix}\otimes\begin{pmatrix}e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2}\\ e^{i\frac{\phi}{2}}\sin\frac{\theta}{2}\end{pmatrix}} \qquad (3)
\end{aligned}
$$

$$
\begin{aligned}
S[\Lambda]\begin{pmatrix}0\\0\\0\\1\end{pmatrix} &= \begin{pmatrix}\cosh\frac{\eta}{2} & 0\\ 0 & \cosh\frac{\eta}{2}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2} & -e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2}\\ e^{i\frac{\phi}{2}}\sin\frac{\theta}{2} & e^{i\frac{\phi}{2}}\cos\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\\
&+\begin{pmatrix}0 & \sinh\frac{\eta}{2}\\ \sinh\frac{\eta}{2} & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2} & e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2}\\ e^{i\frac{\phi}{2}}\sin\frac{\theta}{2} & -e^{i\frac{\phi}{2}}\cos\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\\
&=\begin{pmatrix}0\\ \cosh\frac{\eta}{2}\end{pmatrix}\otimes\begin{pmatrix}-e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2}\\ e^{i\frac{\phi}{2}}\cos\frac{\theta}{2}\end{pmatrix}+\begin{pmatrix}\sinh\frac{\eta}{2}\\ 0\end{pmatrix}\otimes\begin{pmatrix}e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2}\\ -e^{i\frac{\phi}{2}}\cos\frac{\theta}{2}\end{pmatrix}\\
&=\boxed{\begin{pmatrix}-\sinh\frac{\eta}{2}\\ \cosh\frac{\eta}{2}\end{pmatrix}\otimes\begin{pmatrix}-e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2}\\ e^{i\frac{\phi}{2}}\cos\frac{\theta}{2}\end{pmatrix}} \qquad (4)
\end{aligned}
$$

Obteniendo así los resultados deseados.